JOURNAL OF COMBINATORIAL THEORY, Series B 23, 223–226 (1977)

Complementing Permutations in Finite Lattices

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Communicated by the Editors

Received October 31, 1975

In a finite lattice L for which the Möbius function is nonvanishing on all upper and lower intervals, we establish the existence of a permutation of L taking each lattice element to one of its complements.

1. INTRODUCTION

Let L be a finite lattice with zero element 0, unit element 1, and Möbius function μ. Wilson and the author have shown [2] that if

$$\mu(x, 1) \neq 0 \qquad \text{for all} \quad x \in L, \tag{1}$$

there exists a permutation $\sigma: L \to L$ taking each lattice element to an upper semicomplement: $x \vee \sigma(x) = 1$. By duality, if

$$\mu(0, x) \neq 0 \qquad \text{for all} \quad x \in L, \tag{2}$$

there exists a permutation $\tau: L \to L$ taking each element to a lower semicomplement: $x \wedge \tau(x) = 0$. In this paper we show that when both (1) and (2) hold, we may take $\sigma = \tau$, i.e., there exists a permutation $\sigma: L \to L$ sending each element to a complement:[1] $x \vee \sigma(x) = 1$ and $x \wedge \sigma(x) = 0$. We note that properties (1) and (2) are enjoyed by finite geometric lattices and by the lattices of faces of convex polytopes.

2. THE THEOREM

THEOREM. *For any finite lattice L satisfying* (1) *and* (2), *there exists a permutation $\sigma: L \to L$ such that for all $x \in L$,*

$$x \vee \sigma(x) = 1 \qquad \textit{and} \qquad x \wedge \sigma(x) = 0.$$

* Research supported in part by NSF Grant GP-38951 (O.S.U.R.F. Project No. 3684-A1).

[1] The nonvanishing of $\mu(0, 1)$ implies that L is complemented [1].

ISSN 0095-8956

Proof. The proof relies on a vector space argument similar to that of [2]. We consider the vector space V over the rationals $\mathbb{Q}$ freely generated by the elements of L. Formally, we may take V to be the vector space of all maps $L \to \mathbb{Q}$. The standard basis of V is the set $\{I_x \mid x \in L\}$, where

$$I_x(y) = 1 \quad \text{if } y = x, \qquad = 0 \quad \text{otherwise}$$

As in [2], we consider a second set $\{J_x \mid x \in L\}$, given by

$$J_x = \sum_{y: y \vee x = 1} I_y .$$

Using property (1) of L, we show in [2] that

$$I_x = \sum_{y \in L} \lambda(x, y)\, J_y , \tag{3}$$

where

$$\lambda(x, y) = \sum_{a \leqslant x \wedge y} (\mu(a, x)\, \mu(a, y)/\mu(a, 1)).$$

It follows from (3) that $\{J_x \mid x \in L\}$ is a basis of V.

Now we introduce a third subset $\{P_z \mid z \in L\}$ of V, defined by

$$P_z = \sum_{x: x \vee z = 1} \alpha(z, x)\, I_x ,$$

where

$$\alpha(z, x) = \sum_{y: y \leqslant z,\, y \vee x = 1} \mu(0, y). \tag{4}$$

The coefficient $\alpha(z, x)$ is equal to $\mu_f(0, z)$ for the lattice $[0, z]_f$ of Crapo [1], defined with respect to the order-preserving map f: $u \mapsto u \vee x$ of $[0, z]$ into $[x, 1]$. The lattice $[0, z]_f$ consists of z together with all $u \in [0, z]$ for which $u \vee x < 1$, ordered as in L. In the proof of [1, Theorem 3], Crapo shows, using Rota's first crosscut theorem [3], that $\alpha(z, x) = 0$ unless z and x are complements. Letting C denote the complement relation on L: xCy if and only if $x \vee y = 1$, $x \wedge y = 0$; we therefore have

$$\alpha(z, x) \neq 0 \quad \text{implies} \quad zCx. \tag{5}$$

We now show that $\{P_z \mid z \in L\}$ is a basis of V. In view of (5),

$$P_z = \sum_{x: xCz} \alpha(z, x)\, I_x . \tag{6}$$

If xCz and $y \leqslant z$, then $y \vee x = 1$ if and only if yCx, so (4) may be written

$$\alpha(z, x) = \sum_{y:yCx} \mu(0, y)\, \zeta(y, z), \tag{7}$$

where ζ is the zeta function of L. Thus by (6) and (7),

$$P_z = \sum_{x:xCz} I_x \sum_{y:yCx} \mu(0, y)\, \zeta(y, z). \tag{8}$$

Now let $b \in L$ and consider the sum

$$\begin{aligned}\sum_{z:z\leqslant b} \mu(z, b)\, P_z &= \sum_{z:z\leqslant b}\ \sum_{x:xCz} I_x \sum_{y:yCx} \mu(0, y)\, \zeta(y, z)\, \mu(z, b)\\ &= \sum_{x\in L} I_x \sum_{z:zCx}\ \sum_{y:yCx} \mu(0, y)\, \zeta(y, z)\, \mu(z, b)\\ &= \sum_{x\in L} \beta(x, b)\, I_x, \end{aligned} \tag{9}$$

where

$$\beta(x, b) = \sum_{z:zCx}\ \sum_{y:yCx} \mu(0, y)\, \zeta(y, z)\, \mu(z, b). \tag{10}$$

Clearly, $\beta(x, b) = 0$ unless $x \vee b = 1$, while if $b = 1$, the expression in (10) is Crapo's complementation formula [1, Theorem 3] for $\mu(0, 1)$. His argument remains valid, however, for any b such that $x \vee b = 1$, and shows in this case that $\beta(x, b) = \mu(0, b)$. Thus, by (9),

$$\sum_{z:z\leqslant b} \mu(z, b)\, P_z = \sum_{x:x\vee b=1} \mu(0, b)\, I_x = \mu(0, b)\, J_b\,. \tag{11}$$

It now follows from (2), (11), and the fact that $\{J_x \mid x \in L\}$ is a basis of V, that $\{P_z \mid z \in L\}$ is also a basis. Thus the matrix with rows and columns indexed by L, the entry in row z and column x being

$$\begin{aligned} P_z(x) &= \alpha(z, x) \qquad \text{if } zCx,\\ &= 0 \qquad \text{otherwise,}\end{aligned}$$

is nonsingular. Some term in the determinant expansion is therefore nonzero. so for some permutation $\sigma\colon L \to L$, $P_z(\sigma(z)) \neq 0$ for all $z \in L$. Since $P_z(x)$ is zero unless zCx, the proof is complete.

Remark. Conditions (1) and (2) are not necessary for the conclusion of the theorem to hold. Each fails, for example, in the six-element lattice obtained by identifying the zeros and the ones of two four-element chains. They are, however, minimal in the sense that there exist finite complemented lattices for which the Möbius function is nonvanishing on upper intervals and vanishes on a single lower interval (or dually), but which admit no complementing permutation. As an example, let $n \geqslant 3$ and let B be the Boolean lattice of all subsets of an n-set. To the two copoints b, c covering a fixed coline a of B, add a new copoint e covering a but no other element of B. The resulting lattice L is complemented and satisfies (1) and (2) with the exception that $\mu(0, e) = 0$. The L-complements of members of the 4-set $\{a, b, c, e\}$ are contained in the 3-set $\{a', b', c'\}$ of B-complements of $\{a, b, c\}$, so L does not admit a complementing permutation.

REFERENCES

1. H. H. CRAPO, The Möbius function of a lattice, *J. Combinatorial Theory* **1** (1966), 126–131.
2. T. A. DOWLING AND R. M. WILSON, Whitney number inequalities for geometric lattices, *Proc. Amer. Math. Soc.* **47** (1975), 504–512.
3. G.-C. ROTA, On the foundations of combinatorial theory, I: Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **2** (1964), 340–368.